INTEGRABILITY OF INVARIANT METRICS ON THE DIFFEOMORPHISM GROUP OF THE CIRCLE

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Abstract. Each H^k Sobolev inner product ($k \geq 0$) defines a Hamiltonian vector field X_k on the regular dual of the Lie algebra of the diffeomorphism group of the circle. We show that only X_0 and X_1 are bi-Hamiltonian relatively to a modified Lie-Poisson structure.

1. Introduction

Often motions of inertial mechanical systems are described in Lagrangian variables by paths on a configuration space G that is a Lie group. The velocity phase space is the tangent bundle TG and the kinetic energy

$$\mathcal{K} = \frac{1}{2} < v, v >$$

for $v \in TG$. For example, in continuum mechanics the state of a system at time $t \geq 0$ can be specified by a diffeomorphism $x \mapsto \varphi(t,x)$ of the ambient space, giving the configuration of the particles with respect to their initial positions at time $t = 0$. Here x is a label identifying a particle, taken to be the position of the particle at time $t = 0$ so that $\varphi(0,x) = x$. In this setting G would be the group of diffeomorphisms. The material (Lagrangian) velocity field is given by $(t,x) \mapsto \varphi_t(t,x)$ while the spatial (Eulerian) velocity field is $u(t,y) = \varphi_t(t,x)$, where $y = \varphi(t,x)$, i.e. $u = \varphi_t \circ \varphi^{-1}$. Observe that for any fixed time-independent diffeomorphism η, the spatial velocity field $u = \varphi_t \circ \varphi^{-1}$ along the path $t \mapsto \varphi(t)$ remains unchanged if we replace this path by $t \mapsto \varphi(t) \circ \eta$. This right-invariance property suggests to extend the kinetic energy $\mathcal{K}$ by right translation to a right-invariant Lagrangian $\mathcal{K} : TG \to \mathbb{R}$, obtaining a Lagrangian system on G. The length of a path $\{\varphi(t)\}_{t \in [a,b]}$ in G is defined as

$$\mathfrak{l}(\varphi) = \int_a^b < \varphi_t, \varphi_t >^{1/2} \, dt.$$

The Least Action Principle holds if the equation of motion is the geodesic equation. The set $\mathrm{Diff}(\mathbb{S}^1)$ of all smooth orientation-preserving diffeomorphisms of the circle represents the configuration space for the spatially periodic motion of inertial one-dimensional mechanical systems. $\mathrm{Diff}(\mathbb{S}^1)$ is an infinite dimensional Lie group, the group operation being composition [19] and its Lie algebra $\mathrm{Vect}(\mathbb{S}^1)$ being the space of all smooth vector fields on $\mathbb{S}^1$

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cf. [29]. On the regular (or L^2) dual $\mathrm{Vect}^*(\mathbb{S}^1)$ of the Lie algebra $\mathrm{Vect}(\mathbb{S}^1)$ there are some affine canonical Lie-Poisson structures, called *modified Lie-Poisson structures*, which are all compatible. On the other hand, we can consider on the regular dual $\mathrm{Vect}^*(\mathbb{S}^1)$ a countable family $\{X_k\}_{k\geq 0}$ of Hamiltonian vector fields defined by Sobolev inner products. The importance of these inner products lies in that each gives rise via right translation to a geodesic flow on $\mathrm{Diff}(\mathbb{S}^1)$, the Riemannian exponential map of which defines a local chart for every $k \geq 1$ cf. [10] - a property which fails for the Lie group exponential map [19, 27] as well as for the Riemannian exponential map if $k = 0$ [9]. In this paper we show that the Hamiltonian vector field X_k is bi-Hamiltonian relatively to a modified Lie-Poisson structure if and only if $k \in \{0, 1\}$.

2. Preliminaries

In this section, we review some fundamental aspects of finite dimensional smooth Poisson manifolds.

Definition 2.1. A *symplectic manifold* is a pair (M, ω), where M is a manifold and ω is a closed nondegenerate 2-form on M, that is $d\omega = 0$ and for each $m \in M$, $\omega_m : T_mM \times T_mM \to \mathbb{R}$ is a continuous bilinear skew-symmetric map such that the induced linear map $\tilde{\omega}_v : T_mM \to T_m^*M$ defined by $\tilde{\omega}_v(w) = \omega(v, w)$ is an isomorphism for all $v \in T_mM$.

Example 2.2. In the general study of variational problems, extensive use is made of the canonical symplectic structure on the cotangent bundle T^*M (representing the phase space) of the manifold M (representing the configuration space). This symplectic form is given in any local trivialization $(q, p) \in U \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$ of T^*M by

$$\omega_{(q,p)}\Big((Q, P),\, (\tilde{Q}, \tilde{P})\Big) = \tilde{P} \cdot Q - P \cdot \tilde{Q}, \qquad (Q, P),\, (\tilde{Q}, \tilde{P}) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Since a symplectic form ω is nondegenerate, it induces an isomorphism

$$\flat : TM \to T^*M, \quad X \mapsto X^\flat, \tag{2.1}$$

defined via $X^\flat(Y) = \omega(X, Y)$. The *symplectic gradient* X_f of a function f is defined by the relation $X_f^\flat = -df$. The inverse of the isomorphism $\flat$ defines a skew-symmetric bilinear form W on the cotangent space of M. This bilinear form W induces itself a bilinear mapping on $\mathrm{C}^\infty(M)$, the space of smooth functions $f : M \to \mathbb{R}$, given by

$$\{\, f, g\} = W(df, dg) = \omega(X_f, X_g), \quad f, g \in \mathrm{C}^\infty(M), \tag{2.2}$$

and called the *Poisson bracket* of the functions f and g.

Example 2.3. In Example 2.2, the Poisson bracket is given by

$$\{\, f, g\} = \sum_{i=1}^{n} \Big(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \Big). \tag{2.3}$$

The observation that a bracket like (2.3) could be introduced on $\mathrm{C}^\infty(M)$ for a smooth manifold M, without the use of a symplectic form, leads to the general notion of a *Poisson structure* [26].

Definition 2.4. A *Poisson structure* on a C^∞ manifold M is a skew-symmetric bilinear mapping $(f,g) \mapsto \{f,g\}$ on the space $\mathrm{C}^\infty(M)$, which satisfies the *Jacobi identity*

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0, \tag{2.4}$$

as well as the *Leibnitz identity*

$$\{f,gh\} = \{f,g\}\,h + g\,\{f,h\}. \tag{2.5}$$

When the Poisson structure is induced by a symplectic structure ω, the *Leibnitz identity* is a direct consequence of (2.2), whereas the *Jacobi identity* (2.4) corresponds to the condition $d\omega = 0$ satisfied by the symplectic form ω. In the general case, the fact that the mapping $g \mapsto \{f,g\}$ satisfies (2.5) means that it is a derivation of $\mathrm{C}^\infty(M)$. Each derivation on $\mathrm{C}^\infty(M)$ for a C^∞ manifold (even in the infinite dimensional case cf. [1]) corresponds to a smooth vector field, that is, to each $f \in \mathrm{C}^\infty(M)$ is associated a vector field $X_f : M \to TM$, called the *Hamiltonian vector field* of f, such that

$$\{f,g\} = X_f \cdot g = dg.X_f, \tag{2.6}$$

where $dg.X_f = L_{X_f} g$ is the *Lie derivative* of g along X_f. Conversely, a vector field $X : M \to TM$ on a Poisson manifold M is said to be *Hamiltonian* if there exists a function f such that $X = X_f$.

Recall [29] that for a smooth vector field $X : M \to TM$, the Lie derivative operator $L_X : \mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M)$ acts on smooth functions $g : M \to \mathbb{R}$ with differentials $dg : M \to T^*M$ by $(L_X g)(m) = dg(m) \cdot X(m)$ for $m \in M$. The space $\mathrm{Vect}(M)$ of smooth vector fields on M and the space of operators $\{L_X : X \in \mathrm{Vect}(M)\}$ are isomorphic as real vector spaces, the linear isomorphism between them being $X \mapsto L_X$ [1]. Therefore the elements of $\mathrm{Vect}(M)$ can be regarded as operators on $\mathrm{C}^\infty(M)$ via $X \cdot f = L_X f$, forming a Lie algebra if endowed with the bracket $[X,Y] = L_X \circ L_Y - L_Y \circ L_X$. Notice that (2.4) yields

$$[X_f, X_g] = X_{\{f,g\}}. \tag{2.7}$$

From (2.7) it follows (see [29]) that $g \in \mathrm{C}^\infty(M)$ is a constant of motion for X_f if and only if $\{f,g\} = 0$.

Jost [21] pointed out that, just like a derivation on $\mathrm{C}^\infty(M)$ corresponds to a vector field, a bilinear bracket $\{f,g\}$ satisfying the Leibnitz rule (2.5) corresponds to a skew-symmetric bilinear form on TM. That is, there exists a C^∞ tensor field $W \in \Gamma(\bigwedge^2 TM)$, called the *Poisson bivector* of $(M, \{\cdot,\cdot\})$, such that

$$\{f,g\} = W(df, dg).$$

Using the unique local extension of the Lie bracket of vector fields to skew-symmetric multivector fields, called the *Schouten-Nijenhuis bracket* [30], the condition (2.4) becomes

$$[W,W] = 0. \tag{2.8}$$

Conversely, any $W \in \Gamma(\bigwedge^2 TM)$ that satisfies (2.8) induces a Poisson structure on M via (2.2). The only condition that must be satisfied by W is (2.8) since (2.5) holds automatically. A Poisson structure on M is therefore equivalent to a bivector W that satisfies (2.8). This induces a homomorphism

$$\# : T^*M \to TM, \quad \alpha \mapsto \alpha^\#, \tag{2.9}$$

such that $\beta(\alpha^\#) = W(\beta, \alpha)$ for every $\beta \in T^*M$. Notice that for $f \in \mathrm{C}^\infty(M)$ we have $(df)^\# = X_f$. If the homomorphism (2.9) is an isomorphism we call the Poisson structure *nondegenerate*. A nondegenerate Poisson structure on M is equivalent to a symplectic structure where the symplectic form ω is just $\#W$, the closedness condition corresponding to the Jacobi identity [30].

Remark 2.5. The notion of a Poisson manifold is more general than that of a symplectic manifold. For example, in the symplectic case the Poisson bracket satisfies the additional property that $\{ f, g\} = 0$ for all $g \in \mathrm{C}^\infty(M)$ only if $f \in \mathrm{C}^\infty(M)$ is constant, whereas for Poisson manifolds such non-constant functions f might exist, in which case they are called *Casimir functions*. To highlight this, notice that by Darboux' theorem [29] a finite dimensional symplectic manifold M has to be even dimensional and locally there are coordinates $\{q_1, ..., q_n, p_1, ..., p_n\}$ such that $\{ f, g\}$ is given by (2.3). On the other hand, on $M = \mathbb{R}^{2n+1}$ with coordinates $\{q_1, ..., q_n, p_1, ..., p_n, \zeta\}$ we determine a Poisson structure defining the Poisson bracket of $f, g \in C^\infty(\mathbb{R}^{2n+1})$ by the same formula (2.3). Notice that any $f \in C^\infty(\mathbb{R}^{2n+1})$ which depends only on ζ is a Casimir function.

Two Poisson bivectors W_1 and W_2 define *compatible* Poisson structures if

$$[W_1, W_2] = 0. \tag{2.10}$$

This is equivalent to say that for any $\lambda, \mu \in \mathbb{R}$,

$$\{ f, g\}_{\lambda,\mu} = \lambda \{ f, g\}_1 + \mu \{ f, g\}_2$$

is also a Poisson bracket. On a manifold M equipped with two compatible Poisson structures, a vector field X is said to be (formally) *integrable* or *bi-Hamiltonian* if it is Hamiltonian for both structures.

On a symplectic manifold (M, ω), a necessary condition for a vector field X to be Hamiltonian is that $L_X\omega = 0$ [29]. A similar criterion exists for a Poisson manifold (M, W). It is instructive for later considerations to present a short proof of this known result.

Proposition 2.6. *On a Poisson manifold (M, W) a necessary condition for a vector field X to be* Hamiltonian *is*

$$L_X W = 0. \tag{2.11}$$

Proof. If X is Hamiltonian, there is a function $h \in \mathrm{C}^\infty(M)$ such that $X = X_h$. Let f and g be arbitrary smooth functions on M. We have

$$L_X W\,(df, dg) = L_X\,(W(df, dg)) - W\,(L_X df, dg) - W\,(df, L_X dg)\,.$$

But $L_{X_h} f = \{ h, f\}$ and $L_{X_h} df = dL_{X_h} f = d\{ h, f\}$. Therefore

$$\begin{aligned} L_X W(df, dg) &= L_X \{ f, g\} - W\,(d\{ h, f\}, dg) - W\,(df, d\{ h, g\}) \\ &= \{ h, \{ f, g\}\} - \{\{ h, f\}, g\} - \{ f, \{ h, g\}\}\,. \end{aligned}$$

This last expression equals zero because of the Jacobi identity. ∎

The fundamental example of a non-symplectic Poisson structure is the *Lie-Poisson structure* on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$.

Definition 2.7. On the dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ of a Lie group G, there is a Poisson structure defined by

$$\{f, g\}(m) = m([d_m f, d_m g]) \tag{2.12}$$

for $m \in \mathfrak{g}^*$ and $f, g \in C^\infty(\mathfrak{g}^*)$, called the *canonical Lie-Poisson structure* [1].

Remark 2.8. The canonical Lie-Poisson structure has the remarkable property to be *linear*. A Poisson bracket on a vector space is said to be *linear* if the bracket of two linear functionals is itself a linear functional.

Each element $\gamma \in \bigwedge^2 \mathfrak{g}^*$ can be viewed as a Poisson bivector. Indeed, $[\gamma, \gamma] = 0$ since γ is a constant tensor field. As such, γ defines a Poisson structure on $\mathfrak{g}^*$. The condition of compatibility with the canonical Lie-Poisson structure, $[W_0, \gamma] = 0$, can be written as (see [30], Chapter 3)

$$\gamma([u, v], w) + \gamma([v, w], u) + \gamma([w, u], v) = 0, \qquad u, v, w \in \mathfrak{g}. \tag{2.13}$$

On a Lie group G, a right-invariant k-form ω is completely defined by its value at the unit element e, and hence by an element of $\bigwedge^k \mathfrak{g}^*$. In other words, there is a natural isomorphism between the space of right-invariant k-forms on G and $\bigwedge^k \mathfrak{g}^*$. Moreover, since the exterior differential d commutes with right translations, it induces a linear operator $\partial : \bigwedge^k \mathfrak{g}^* \to \bigwedge^{k+1} \mathfrak{g}^*$ that satisfies $\partial \circ \partial = 0$ and

(1) $\partial\gamma = 0$ for $\gamma \in \bigwedge^0 \mathfrak{g}^* = \mathbb{R}$;
(2) $\partial\gamma(u, v) = -\gamma([u, v])$ for $\gamma \in \bigwedge^1 \mathfrak{g}^* = \mathfrak{g}^*$;
(3) $\partial\gamma(u, v, w) = \gamma([u, v], w) + \gamma([v, w], u) + \gamma([w, u], v)$ for $\gamma \in \bigwedge^2 \mathfrak{g}^*$,

where $u, v, w \in \mathfrak{g}$, as one can check by direct computation (see [18], Chapter 24). The kernel $Z^n(\mathfrak{g})$ of $\partial : \bigwedge^n(\mathfrak{g}^*) \to \bigwedge^{n+1}(\mathfrak{g}^*)$ is the space of n-*cocycles* and the range $B^n(\mathfrak{g})$ of $\partial : \bigwedge^{n-1}(\mathfrak{g}^*) \to \bigwedge^n(\mathfrak{g}^*)$ is the spaces of n-*coboundaries*. Notice that $B^n(\mathfrak{g}) \subset Z^n(\mathfrak{g})$. The quotient space $H^n_{CE}(\mathfrak{g}) = Z^n(\mathfrak{g})/B^n(\mathfrak{g})$ is the n-th *Lie algebra cohomology* or *Chevaley-Eilenberg cohomology group* of $\mathfrak{g}$. Notice that in general the Lie algebra cohomology is different from the de Rham cohomology H^n_{DR}. For example, $H^1_{DR}(\mathbb{R}) = \mathbb{R}$ but $H^1_{CE}(\mathbb{R}) = 0$.

Each 2-cocycle γ defines a Poisson structure on $\mathfrak{g}^*$ compatible with the canonical one. Indeed (2.13) can be recast as $\partial\gamma = 0$. Notice that the Hamiltonian vector field X_f of a function $f \in C^\infty(\mathfrak{g}^*)$ computed with respect to the Poisson structure defined by the 2-cocycle γ is

$$X_f(m) = \gamma(d_m f, \cdot). \tag{2.14}$$

Definition 2.9. A *modified Lie-Poisson structure* is a Poisson structure on $\mathfrak{g}^*$ whose Poisson bivector is given by $W_\gamma = W_0 + \gamma$, where W_0 is the canonical Poisson bivector and γ is a 2-cocycle.

[1]Here, $d_m f$, the differential of a function $f \in C^\infty(\mathfrak{g}^*)$ at $m \in \mathfrak{g}^*$ is to be understood as an element of the Lie algebra $\mathfrak{g}$

Example 2.10. A special case of modified Lie-Poisson structure is given by a 2-cocycle γ which is a coboundary. If $\gamma = \partial m_0$ for some $m_0 \in \mathfrak{g}^*$, the expression

$$\{ f, g\}_\gamma (m) = m_0([d_m f, d_m g])$$

looks like if the Lie-Poisson bracket had been "frozen" at a point $m_0 \in \mathfrak{g}^*$ and for this reason some authors call it a *"freezing"* structure.

3. Modified Lie-Poisson structures on $\mathrm{Vect}(\mathbb{S}^1)$

The group $\mathrm{Diff}(\mathbb{S}^1)$ of smooth orientation-preserving diffeomorphisms of the circle $\mathbb{S}^1$ is endowed with a smooth manifold structure based on the Fréchet space $\mathrm{C}^\infty(\mathbb{S}^1)$. The composition and the inverse are both smooth maps $\mathrm{Diff}(\mathbb{S}^1) \times \mathrm{Diff}(\mathbb{S}^1) \to \mathrm{Diff}(\mathbb{S}^1)$, respectively $\mathrm{Diff}(\mathbb{S}^1) \to \mathrm{Diff}(\mathbb{S}^1)$, so that $\mathrm{Diff}(\mathbb{S}^1)$ is a Lie group [19]. Its Lie algebra $\mathrm{Vect}(\mathbb{S}^1)$ is the space of smooth vector fields on $\mathbb{S}^1$, which is isomorphic to the space $\mathrm{C}^\infty(\mathbb{S}^1)$ of periodic functions. The Lie bracket on $\mathrm{Vect}(\mathbb{S}^1)$ is given by

$$[u, v] = uv_x - u_x v.$$

Since the topological dual of the Fréchet space $\mathrm{Vect}(\mathbb{S}^1)$ is too big, being isomorphic to the space of distributions on the circle, we restrict our attention in the following to the *regular dual* $\mathrm{Vect}^*(\mathbb{S}^1)$, the subspace of distributions defined by linear functionals of the form

$$u \mapsto \int_{\mathbb{S}^1} mu\, dx$$

for some function $m \in \mathrm{C}^\infty(\mathbb{S}^1)$. The regular dual $\mathrm{Vect}^*(\mathbb{S}^1)$ is therefore isomorphic to $\mathrm{C}^\infty(\mathbb{S}^1)$ by means of the L^2 inner product [2]

$$< u, v >= \int_{\mathbb{S}^1} uv\, dx.$$

Let f be a smooth real valued function on $\mathrm{C}^\infty(\mathbb{S}^1)$. Its *Fréchet* derivative at m, $df(m)$ is a linear functional on $\mathrm{C}^\infty(\mathbb{S}^1)$. We say that f is a *regular function* if there exists a smooth map $\delta f : \mathrm{C}^\infty(\mathbb{S}^1) \to \mathrm{C}^\infty(\mathbb{S}^1)$ such that

$$df(m)\, M = \int_{\mathbb{S}^1} M \cdot \delta f(m)\, dx, \qquad m, M \in \mathrm{C}^\infty(\mathbb{S}^1).$$

That is, the Fréchet derivative $df(m)$ belongs to the regular dual $\mathrm{Vect}^*(\mathbb{S}^1)$ and the mapping $m \mapsto \delta f(m)$ is smooth. The map δf is a vector field on $\mathrm{C}^\infty(\mathbb{S}^1)$, called the *gradient* of f for the L^2-metric. In other words, a regular function is a smooth function on $\mathrm{C}^\infty(\mathbb{S}^1)$ which has a smooth gradient.

Example 3.1. Typical examples of *regular functions* are nonlinear *functionals* over the space $\mathrm{C}^\infty(\mathbb{S}^1)$, like

$$f(m) = \int_{\mathbb{S}^1} \left(m^2 + mm_x^2\right)\, dx \quad \text{with} \quad \delta f(m) = 2m - m_x^2 - 2mm_{xx},$$

as well as linear functionals

$$f(m) = \int_{\mathbb{S}^1} um\, dx \quad \text{with} \quad \delta f(m) = u \in \mathrm{C}^\infty(\mathbb{S}^1).$$

[2]In the sequel, we use the notation $u, v, \dots$ for elements of $\mathrm{Vect}(\mathbb{S}^1)$ and $m, n, \dots$ for elements of $\mathrm{Vect}^*(\mathbb{S}^1)$ to distinguish them, although they all belong to $\mathrm{C}^\infty(\mathbb{S}^1)$.

Notice that the smooth function $f_\theta : \mathrm{C}^\infty(\mathbb{S}^1) \to \mathbb{R}$ defined by $f_\theta(m) = m(\theta)$ for some fixed $\theta \in \mathbb{S}^1$ is not regular as δf_θ is the Dirac measure at θ.

Conversely, a smooth vector field X on $\mathrm{Vect}^*(\mathbb{S}^1)$ is called a *gradient* if there exists a *regular function* f on $\mathrm{Vect}^*(\mathbb{S}^1)$ such that $X(m) = \delta f(m)$ for all $m \in \mathrm{Vect}^*(\mathbb{S}^1)$. Observe that if f is a smooth real valued function on $\mathrm{C}^\infty(\mathbb{S}^1)$ then its second Fréchet derivative is symmetric [19], that is,

$$d^2 f(m)(M,N) = d^2 f(m)(N,M), \qquad m, M, N \in \mathrm{C}^\infty(\mathbb{S}^1).$$

For a regular function, this property can be written as

$$\int_{\mathbb{S}^1} \Big(d\,\delta f(m)M\Big) N\, dx = \int_{\mathbb{S}^1} \Big(d\,\delta f(m)N\Big) M\, dx, \tag{3.1}$$

for all $m, M, N \in \mathrm{C}^\infty(\mathbb{S}^1)$. Hence the linear operator $d\,\delta f(m)$ is symmetric for the L^2-inner product on $\mathrm{C}^\infty(\mathbb{S}^1)$ for each $m \in \mathrm{C}^\infty(\mathbb{S}^1)$. We will resume this fact in the following lemma.

Lemma 3.2. *A necessary condition for a vector field X on $\mathrm{C}^\infty(\mathbb{S}^1)$ to be a* gradient *is that its Fréchet derivative $dX(m)$ is a symmetric linear operator.*

To define a *Poisson bracket* on the space of *regular functions* on $\mathrm{Vect}^*(\mathbb{S}^1)$, we consider a one-parameter family of linear operators $J(m)$ and set

$$\{\, f, g\}\,(m) = \int_{\mathbb{S}^1} \delta f(m)\, J(m)\, \delta g(m)\, dx. \tag{3.2}$$

The operators $J(m)$ must satisfy certain conditions in order for (3.2) to be a valid Poisson structure on $\mathrm{Vect}^*(\mathbb{S}^1)$.

Definition 3.3. A family of linear operators $J(m)$ on $\mathrm{Vect}^*(\mathbb{S}^1)$ defines a Poisson structure on $\mathrm{Vect}^*(\mathbb{S}^1)$ if (3.2) satisfies

(1) $\{\, f, g\}$ is regular if f and g are regular,
(2) $\{\, g, f\} = -\{\, f, g\}$,
(3) $\{\,\{\, f, g\}\,, h\} + \{\,\{\, g, h\}\,, f\} + \{\,\{\, h, f\}\,, g\} = 0$.

Notice that the second condition above simply means that $J(m)$ is a skew-symmetric operator for each m.

Example 3.4. The canonical Lie-Poisson structure on $\mathrm{Vect}^*(\mathbb{S}^1)$ given by

$$\{\, f, g\}\,(m) = m\,([\delta f, \delta g]) = \int_{\mathbb{S}^1} \delta f(m)\,(mD + Dm)\,\delta g(m)\, dx$$

is represented by the one-parameter family of skew-symmetric operators

$$J(m) = mD + Dm \tag{3.3}$$

where $D = \partial_x$. It can be checked that all the three required properties are satisfied. In particular, we have

$$\delta\,\{\, f, g\} = d\,\delta f(J\delta g) - d\,\delta g(J\delta f) + \delta f\,\delta g_x - \delta g\,\delta f_x.$$

Definition 3.5. The *Hamiltonian* of a *regular* function f, for a Poisson structure defined by J is defined as the vector field

$$X_f(m) = J(m)\,\delta f(m).$$

Proposition 3.6. *A necessary condition for a smooth vector field X on* $\mathrm{Vect}^*(\mathbb{S}^1)$ *to be* Hamiltonian *with respect to the Poisson structure defined by a* constant *linear operator K is the symmetry of the operator $dX(m) \circ K$ for each $m \in \mathrm{Vect}^*(\mathbb{S}^1)$.*

Proof. If X is Hamiltonian, we can find a regular function f such that

$$X(m) = K\delta f(m).$$

Moreover, since K is a constant linear operator, we have

$$d\big(K\,\delta f\big)(m)\,M = K \circ \big(d\delta f(m)\big)\,M.$$

Therefore, we get

$$\begin{aligned} < dX(m) \circ K\,M,\ N > &=< K \circ d\delta f(m) \circ K\,M,\ N > \\ &=< M,\ K \circ d\delta f(m) \circ K\,N > \\ &=< M,\ dX(m) \circ K\,N >, \end{aligned}$$

since K is skew-symmetric and $d\delta f(m)$ is symmetric. □

A 2-cocycle on $\mathrm{Vect}(\mathbb{S}^1)$ is a bilinear functional γ represented by a skew-symmetric operator $K : \mathrm{C}^\infty(\mathbb{S}^1) \to \mathrm{C}^\infty(\mathbb{S}^1)$ such that

$$\gamma(u, v) =< u, Kv >= \int_{\mathbb{S}^1} u\,K\,v\,dx,$$

and satisfying the Jacobi identity

$$< [u, v]\,, Kw > + < [v, w]\,, Ku > + < [w, u]\,, Kv >= 0.$$

If K is a differential operator we call γ a *differential cocycle*. Gelfand and Fuks [16] observed that all differential 2-cocycles of $\mathrm{Vect}(\mathbb{S}^1)$ belong to the one-dimensional cohomology class generated by $[D^3]$. Moreover, each regular 2-coboundary is represented by the skew-symmetric operator

$$m_0 D + D m_0,$$

for some $m_0 \in \mathrm{C}^\infty(\mathbb{S}^1)$. Therefore, each differential 2-cocycle of $\mathrm{Vect}(\mathbb{S}^1)$ is represented by an operator of the form

$$K = m_0 D + D m_0 + \beta D^3 \tag{3.4}$$

where $m_0 \in \mathrm{C}^\infty(\mathbb{S}^1)$ and $\beta \in \mathbb{R}$ (see also [17]).

For $k \geq 0$ and $u, v \in \mathrm{Vect}(\mathbb{S}^1) \equiv \mathrm{C}^\infty(\mathbb{S}^1)$, let us now define the H^k (Sobolev) inner product by

$$< u,\ v >_k = \int_{\mathbb{S}^1} \sum_{i=0}^{k} (\partial_x^i u)\,(\partial_x^i v)\,dx = \int_{\mathbb{S}^1} A_k(u)\,v\,dx\,,$$

where

$$A_k = 1 - \frac{d^2}{dx^2} + \ldots + (-1)^k \frac{d^{2k}}{dx^{2k}} \tag{3.5}$$

is a continuous linear isomorphism of $\mathrm{C}^\infty(\mathbb{S}^1)$. Note that A_k is a symmetric operator for the L^2 inner product since

$$\int_{\mathbb{S}^1} A_k(u)\,v\,dx = \int_{\mathbb{S}^1} u\,A_k(v)\,dx.$$

The operator A_k gives rise to a Hamiltonian function on $\mathrm{Vect}^*(\mathbb{S}^1)$ given by

$$h_k(m) = \int_{\mathbb{S}^1} \tfrac{1}{2}\, m(A_k^{-1}m)\, dx.$$

The corresponding Hamiltonian vector field X_k is given by

$$X_k(m) = (mD + Dm)(A_k^{-1}m) = 2mu_x + um_x,$$

if we let $m = A_k u$.

Theorem 3.7. *The Hamiltonian vector field X_k is bi-Hamiltonian relatively to a modified Lie-Poisson structure if and only if $k \in \{0, 1\}$.*

Proof. It is well known (see [28]) that X_0 is bi-Hamiltonian with respect to the operator D which represents a coboundary. It is also known that X_1 is a bi-Hamiltonian vector field with respect to the cocycle represented by the operator $D(1 - D^2)$ cf. [2, 11, 14]. Notice that this cocycle is not a coboundary.

We will now show that there is no differential cocycle

$$K = m_0 D + Dm_0 + \beta D^3$$

for which X_k could be Hamiltonian unless $k \in \{0, 1\}$. We have

$$dX_k(m) = 2u_x I + uD + 2mDA_k^{-1} + m_x A_k^{-1},$$

and in particular, for $m = 1$,

$$dX_k(1) = D + 2DA_k^{-1}.$$

Letting

$$P(m) = dX_k(m) \circ K,$$

we obtain that

$$P(1) = \big(D + 2DA_k^{-1}\big) \circ \big(m_0 D + Dm_0\big) + \beta D^4 (1 + 2A_k^{-1}),$$

whereas

$$P(1)^* = \big(m_0 D + Dm_0\big) \circ \big(D + 2DA_k^{-1}\big) + \beta D^4 (1 + 2A_k^{-1}).$$

Therefore, denoting $m_0' = \partial_x m_0$, we have

$$\begin{aligned} P(1) - P(1)^* = \big(m_0' D + Dm_0'\big) + 2\big(A_k^{-1} Dm_0 D - Dm_0 DA_k^{-1}\big) + \\ + 2\big(A_k^{-1} D^2 m_0 - m_0 D^2 A_k^{-1}\big). \end{aligned}$$

If this operator is zero, we must have in particular the relation

$$A_k\big(P(1) - P(1)^*\big) A_k(e^{irx}) = 0,$$

for all $r \in \mathbb{Z}$. But, for $r \neq \pm 1$,

$$A_k(e^{irx}) = f_k(r)\, e^{irx} \quad \text{with} \quad f_k(r) = \frac{r^{2k+2} - 1}{r^2 - 1},$$

and

$$A_k\big(P(1) - P(1)^*\big) A_k(e^{irx})$$

is of the form e^{irx} times a polynomial expression in r with highest order term $2i\, m_0'(x)\, r^{4k+1}$. Therefore, a necessary condition for X_k to be Hamiltonian relatively to the Poisson operator K defined by (3.4) is that m_0 is a constant.

Let $\alpha = 2m_0 \in \mathbb{R}$. Then

$$P(m) = dX_k(m) \circ K = \alpha \left\{ 2u_x D + uD^2 + 2mD^2 A_k^{-1} + m_x D A_k^{-1} \right\} + \\ + \beta \left\{ 2u_x D^3 + uD^4 + 2mD^4 A_k^{-1} + m_x D^3 A_k^{-1} \right\}$$

because D and A_k commute. By virtue of Proposition 3.6, a necessary condition for X_k to be Hamiltonian with respect to the cocycle represented by K is that $P(m)$ is symmetric. That is

$$< P(m)M, N >=< M, P(m)N >, \tag{3.6}$$

for all $m, M, N \in \mathrm{C}^\infty(\mathbb{S}^1)$. Since this last expression is tri-linear in the variables m, M, N, the equality can be checked for complex periodic functions m, M, N where the L^2 inner product is extended naturally into a complex bilinear functional. That is, the extension is not a hermitian product, we just allow homogeneity with respect to the complex scalar field in both components. Let $m = A_k u$, $u = \exp(iax)$, $M = \exp(ibx)$ and $N = \exp(icx)$ with $a, b, c \in \mathbb{Z}$. We have

$$< P(m)M, N >= \Big[(2ab^3 + b^4)\beta - (2ab + b^2)\alpha + \\ + \Big((ab^3 + 2b^4)\beta - (ab + 2b^2)\alpha \Big) \frac{f_k(a)}{f_k(b)} \Big] \int_{\mathbb{S}^1} e^{i(a+b+c)x} dx \,,$$

whereas

$$< M, P(m)N >= \Big[(2ac^3 + c^4)\beta - (2ac + c^2)\alpha + \\ + \Big((ac^3 + 2c^4)\beta - (ac + 2c^2)\alpha \Big) \frac{f_k(a)}{f_k(c)} \Big] \int_{\mathbb{S}^1} e^{i(a+b+c)x} dx \,.$$

For $a = n$, $b = -2n$ and $c = n$, we obtain

$$< P(m)M, N >= (24n^4\beta - 6n^2\alpha) \frac{f_k(n)}{f_k(2n)}, \;\; < M, P(m)N >= 6n^4\beta - 6n^2\alpha. \tag{3.7}$$

The equality of the two expressions in (3.7) for all $n \in \mathbb{N}$ is ensured by means of (3.6). For $k = 1$ this leads to the condition $\alpha + \beta = 0$ and we recover the second Poisson structure given by $K = D - D^3$ for which X_1 is known to be Hamiltonian with Hamiltonian function

$$\tilde{h}_1(m) = \frac{1}{2} \int_{\mathbb{S}^1} \Big((A_1^{-1} m)^3 + (A_1^{-1} m) \left[(A_1^{-1} m)_x \right]^2 \Big) \, dx.$$

In the general case, if $\beta \neq 0$, the leading term with respect to n in the first expression in (3.7) is $(-48\,\beta\, 2^{-2k})$, whereas in the second it is $(-12\,\beta)$. Thus unless $\beta = 0$ we must have $k = 1$. On the other hand, if $\beta = 0$, from (3.6)-(3.7) we infer that $\alpha f_k(n) = \alpha f_k(2n)$ for all $n \in \mathbb{N}$. Thus $\alpha = 0$ unless $k = 0$. For $k = 0$ we recover the Poisson structure given by $K = D$ for which X_0 is Hamiltonian with Hamiltonian function

$$\tilde{h}_0(m) = \frac{1}{2} \int_{\mathbb{S}^1} m^3 \, dx.$$

This completes the proof. □

4. Conclusion

We showed that among all H^k Sobolev inner products on $C^\infty(\mathbb{S}^1)$, only for $k \in \{0,1\}$ is the associated vector field bi-Hamiltonian relatively to a modified Lie-Poisson structure. Endowing $\mathrm{Diff}(\mathbb{S}^1)$ with the H^1 right-invariant metric, the associated geodesic equation turns out to be the Camassa-Holm equation [23] (see also [22])

$$u_t + uu_x + \partial_x(1-\partial_x^2)^{-1}(u^2 + \frac{1}{2}\,u_x^2) = 0,$$

a model for shallow water waves (see [2] and the alternative derivations in [5, 13, 15, 20]) that accommodates waves that exist indefinitely in time [3, 7] as well as breaking waves [6, 8]. The bi-Hamiltonian structure is reflected in the existence of infinitely many conserved integrals for the equation [2, 11, 14, 24] which are very useful in the qualitative analysis of its solutions. Both global existence results and blow-up results can be obtained using certain conservation laws [3, 7, 31], while the proof of stability of traveling wave solutions relies on the specific form of some conserved quantities [4, 11, 12, 25]. On the other hand, the geodesic equation on $\mathrm{Diff}(\mathbb{S}^1)$ for the L^2 right-invariant metric is the inviscid Burgers equation

$$u_t + 3uu_x = 0.$$

This model of gas dynamics has been thoroughly studied (see [9] and references therein). In contrast to the case of the H^1 right-invariant metric [10], the Riemannian exponential map is not a C^1 local diffeomorphism in the case of the L^2 right-invariant metric [9]. This means that of the two bi-Hamiltonian vector fields X_0 and X_1, the second generates a flow on $\mathrm{Diff}(\mathbb{S}^1)$ with properties that parallel those of geodesic flows on finite-dimensional Lie groups.

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